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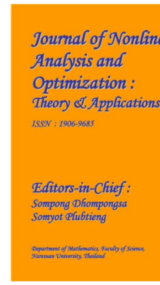

HARMONIC INDEX AND ECCENTRIC HARMONIC INDEX OF STARBARBELL GRAPH AND WHEEL BARBELL GRAPH

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ABSTRACT:
The harmonic index $H(G)$ of a graph G is defined as $\frac{2}{d(u)+d(v)}$ for all edges uv, where $d(u)$ and $d(v)$ be the degree of the vertices u and v respectively. Likewise, the eccentric harmonic index $H_e(G)$ of a graph G is defined as $\frac{2}{e(u)+e(v)}$ for edges uv, where $e(u)$ and $e(v)$ be the eccentricity of the vertices u and v, respectively, of a graph G. Here, we deduce $H(G)$ and $H_e(G)$ for starbarbell graph, wheelbarbell graph and wheel graph.
Keywords: Harmonic index, eccentric harmonic index, starbarbell graph and wheelbarbell graph.
MSC:05C07, 05C12, 05C09.

INTRODUCTION:

In this paper, $G = (V, E)$ be simple finite undirected connected graph with vertex set V and edge set E. Let $|V(G)| = \nu$ and $|E(G)| = \epsilon$, be the order and size of the graph G. For every vertex $v \in V$, the open neighbourhood $N(v)$, is the set of all vertices that are adjacent to v. The degree of vertex v in a graph G is $d(v) = |N(v)|$. A vertex of degree 1 is called pendant vertex. The eccentricity of a vertex u in a graph G is $e(u) = max\{d(u,v): v \in V(G)\}$. A complete graph is a graph in which every pair of vertices is connected by an edge. A complete graph with m vertices is denoted as K_m. A $W_m, m \geq 4$ denote the wheel on m vertices, in which a vertex is joined to all the vertices of a cycle C_{m-1}, see Figure 1.

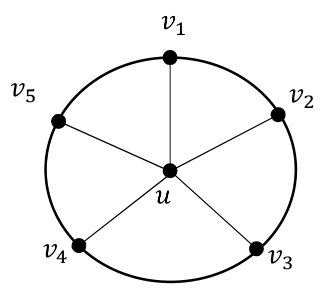

FIGURE 1.W_6- Wheel on 6 vertices.

For $m \geq 2$,S_m denotes the star on m vertices in which one vertex is adjacent to all other remaining $m-1$ vertices. Also $S_m \cong K_{1,m-1}$, see Figure 2.

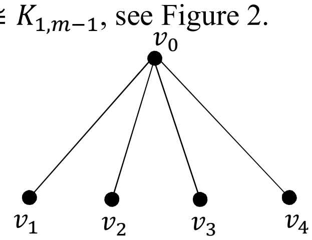

FIGURE 2. S_5- Star on 5 vertices.

A topological index is a numerical quantity related to a graph. In a theoretical chemistry, topological indices is called as molecular descriptor. It is used to analyse and investigate some physicochemical properties of a molecule. Molecular descriptors play a significant role in Quantitative Structure-Property Relationship (QSPR). Also in Quantitative Structure-Activity Relationship (QSAR) investigations. At present, there are numerous topological indices which are classified by the structural properties of the graphs. Furthermore, the most widely used topological indice is the Wiener index [14] which was used by Harold Wiener in 1947, to compare the boiling points of some of the alkane isomers. For a graph G the Wiener index is defined by

$$W(G) = \sum_{u,v \in V(G)} d(u,v),$$

where $d(u,v)$ is the distance between u and v in G and the sum goes over all the unordered pairs of vertices. In literature Randić connectivity index and the Zagreb group indices are calculated using the degree of vertices. In [11] Randić connectivity index, $R(G)$ introduced by Milan Randić given by

$$R(G) = \sum_{uv \in E(G)} \big(d(u)d(v)\big)^{-\frac{1}{2}}.$$

Based on this, a numerous topological indices have been proposed. In [2,3] the first Zagreb index M_1 and the second Zagreb index M_2 are introduced as an successor of R, given by

$$M_1(G) = \sum_{i=1}^{n} d(v_i)^2 \text{ and } M_2(G) = \sum_{uv \in E(G)} d(u)d(v).$$

In [4], Fajtlowicz introduced an another topological index named harmonic index, given by

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d(u) + d(v)}.$$

In [12]Sowaity et.al., had introduced the eccentric harmonic index as similar to harmonic index, by considering the eccentricity of the vertices instead of degree of the vertices, which is given by,

$$H_e(G) = \sum_{uv \in E(G)} \frac{2}{e(u) + e(v)}.$$

Various indices are studied by [8,10].In [16] Zhong determined the minimum and maximum values of harmonic index on unicyclic graphs and characterized the corresponding extremal graphs. Jianxi Li et.al., [5] provide a simpler method for determining the unicyclic graphs with maximum and minimum harmonic index among all unicyclic graphs. Onagh [9] determine harmonic index of cartesian, lexicographic, tensor, strong, corona and edge corona product of two connected graphs. Mahdieh Azari [7] determine the eccentric harmonic index of various families of graph products. Kamel Jebreen et.al., [6] determine eccentric harmonic index for the cartesian product of some particular graphs.

STARBARBELL GRAPH:

In this section we deduce the harmonic index and the eccentric harmonic index of a starbarbell graph In [13] the starbarbell graph $\mathrm{SB}_{m_1,m_2,\dots,m_{n-1}}$ is introduced and it is obtained from star $\mathrm{S_n}$ and $(n-1)$ complete graph K_{m_i} by merging one vertex from each K_{m_i} and the $\mathrm{i^{th}}$ leaf of $\mathrm{S_n}$, where $m_i \geq 2, 1 \leq i \leq n-1$ and $n \geq 3$ (see Figure 3).

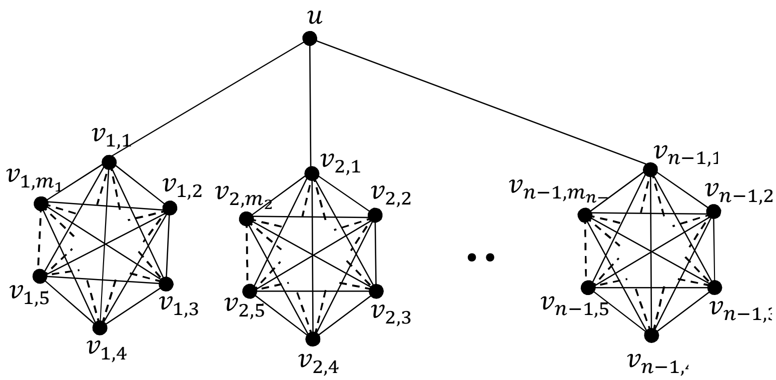

FIGURE 3. Starbarbell graph $SB_{m_1,m_2,\ldots,m_{n-1}}$.

Here we discuss about the starbarbell graph $\mathrm{SB}_{m_1,m_2,\ldots,m_{n-1}}$ with uniform size of the complete graph, that is, $m_1 = m_2 = \cdots = m_{n-1} = m$, $\mathrm{SB}_{\underbrace{m,m,\ldots,m}_{(n-1)\text{times}}}$.

Theorem 2.1. For $n \geq 3$ and $m \geq 2$, the starbarbell graph $G = \mathrm{SB}_{\underbrace{m,m,\ldots,m}_{(n-1)\text{times}}}$ has

(i) $H(G) = (n-1)\left[\frac{(m-2)}{2} + \frac{2(m-1)}{2m-1} + \frac{2}{m+n-1}\right]$

(ii) $H_e(G) = (n-1)\left[\frac{(m-1)(m-2)}{8} + \frac{2(m-1)}{7} + \frac{2}{5}\right]$.

Proof. Let n be the number of vertices of S_n and let m be the number of vertices of complete graph K_m in the starbarbell graph. The starbarbell graph $G = \mathrm{SB}_{\underbrace{m,m,\ldots,m}_{(n-1)\text{times}}}$ has $(n-1)m+1$ vertices and $(n-1)\binom{m}{2} + (n-1)$ edges for $n \geq 3$ and $m \geq 2$. Let $V(G) = \{u\} \cup V_1 \cup V_2 \cdots \cup V_{n-1}$ where $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,m}\}$, for $i = 1, 2, \ldots, n-1$.

(i) The harmonic index of starbarbell graph is

$$H(G) = \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=2}^{m}\sum_{k=2,k\neq j}^{m}\frac{2}{d(v_{i,j}) + d(v_{i,k})} + \sum_{i=1}^{n-1}\sum_{j=2}^{m}\frac{2}{d(v_{i,1}) + d(v_{i,j})} + \sum_{i=1}^{n-1}\frac{2}{d(u) + d(v_{i,1})}.$$

$$= \frac{(n-1)(m-1)(m-2)}{2}\left(\frac{2}{m-1+m-1}\right) + (n-1)(m-1)\left(\frac{2}{m+m-1}\right) + (n-1)\left(\frac{2}{n-1+m}\right).$$

$$= (n-1)\left[\frac{m-2}{2} + \frac{2(m-1)}{2m-1} + \frac{2}{m+n-1}\right].$$

(ii) The eccentricity for the vertices in the graph G is given by

$$e(u) = 2,$$

$e(v_{i,1}) = 3$ for $i = 1, 2, \ldots, n-1$ and

$e(v_{i,j}) = 4$ for $i = 1, 2, \ldots, n-1$ and $j = 2, 3, \ldots, m$.

The eccentric harmonic index of starbarbell graph is

$$H_e(G) = \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=2}^{m}\sum_{k=2,k\neq j}^{m}\frac{2}{e(v_{i,j}) + e(v_{i,k})} + \sum_{i=1}^{n-1}\sum_{j=2}^{m}\frac{2}{e(v_{i,1}) + e(v_{i,j})} + \sum_{i=1}^{n-1}\frac{2}{e(u) + e(v_{i,1})}.$$

$$= \frac{(n-1)(m-1)(m-2)}{2}\left(\frac{2}{4+4}\right) + (n-1)(m-1)\left(\frac{2}{3+4}\right) + (n-1)\left(\frac{2}{2+3}\right).$$

$= (n-1)\left[\frac{(m-1)(m-2)}{8} + \frac{2(m-1)}{7} + \frac{2}{5}\right]$.□

Remark 2.1. In Theorem 2.1 if we take $\mathrm{m} = 1$, then $\mathrm{G} = \mathrm{SB}_{\underbrace{\mathrm{m,m,\dots,m}}_{(\mathrm{n}-1)\mathrm{times}}}$ becomes $\mathrm{G} = \mathrm{SB}_{\underbrace{1,1,\dots,1}_{(\mathrm{n}-1)\mathrm{times}}}$, here $\mathrm{G} \cong \mathrm{S_n}$. From (i) of Theorem 2.1

$$\mathrm{H(G)} = (\mathrm{n}-1)\binom{\mathrm{m}-1}{2}\left(\frac{2}{2(\mathrm{m}-1)}\right) + (\mathrm{n}-1)(\mathrm{m}-1)\left(\frac{2}{\mathrm{m}+\mathrm{m}-1}\right) + (\mathrm{n}-1)\left(\frac{2}{\mathrm{m}+\mathrm{n}-1}\right).$$

If we put $\mathrm{m} = 1$ we have $H(G) = \frac{2(n-1)}{n}$. In [15] they found separately $\mathrm{H(S_n)} = \frac{2(\mathrm{n}-1)}{\mathrm{n}}$. Infact it is a particular case of Theorem 2.1.□

WHEELBARBELL GRAPH:

In this section we discuss some results on wheelbarbell graph for topological indices like harmonic index and eccentric harmonic index.

In [1] the wheelbarbell graph $\mathrm{WB}_{\mathrm{m_1,m_2,\dots,m_{n-1}}}$ is introduced and it is obtained from wheel graph $\mathrm{W_n}$ and $(\mathrm{n}-1)$ complete graph $\mathrm{K_{m_i}}$ by merging one vertex from each $\mathrm{K_{m_i}}$ and the $\mathrm{i^{th}}$ vertex of $\mathrm{W_n}$, where $\mathrm{m_i} \geq 2, 1 \leq \mathrm{i} \leq \mathrm{n}-1$ and $\mathrm{n} \geq 4$, (see Figure 4).

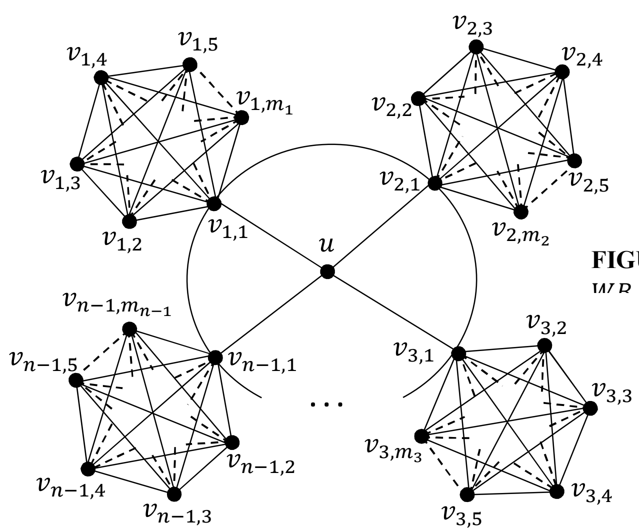

FIGURE 4. Wheelbarbell graph WB

Here we discuss about the wheelbarbell graph with uniform size of the complete graph, that is,

$$m_1 = m_2 = \cdots = m_{n-1} = m, \qquad WB_{\underbrace{m,m,\dots,m}_{(n-1)times}}.$$

Theorem 3.1. Let $\mathrm{G} = \mathrm{WB}_{\underbrace{\mathrm{m,m,\dots,m}}_{(\mathrm{n}-1)\mathrm{times}}}$ be a wheelbarbell graph with $\mathrm{n} \geq 4$ and $\mathrm{m} \geq 2$; then,

(i) $\mathrm{H(G)} = (\mathrm{n}-1)\left[\frac{(\mathrm{m}-2)}{2} + \frac{2(\mathrm{m}-1)}{2\mathrm{m}+1} + \frac{1}{\mathrm{m}+2} + \frac{2}{\mathrm{m}+\mathrm{n}+1}\right]$

(ii) $H_e(G) = \begin{cases} \frac{(m-1)(m-2)}{2} + \frac{6(m-1)}{5} + 3 & : n = 4 \\ (n-1)\left[\frac{(m-1)(m-2)}{8} + \frac{2(m-1)}{7} + \frac{11}{15}\right] & : n \geq 5 \end{cases}$

Proof. Let n be the number of vertices of $\mathrm{W_n}$ and m be the number of vertices of complete graph $\mathrm{K_m}$ in the wheelbarbell graph. The wheelbarbell graph $\mathrm{G} = \mathrm{WB}_{\underbrace{\mathrm{m,m,\dots,m}}_{(\mathrm{n}-1)\mathrm{times}}}$ has $(\mathrm{n}-1)\mathrm{m}+1$ vertices and

$2(\mathrm{n}-1)+(\mathrm{n}-1)\binom{\mathrm{m}}{2}$edges. Let$\mathrm{V(G)}=\{\mathrm{u}\}\cup \mathrm{V}_1\cup \mathrm{V}_2\cdots\cup \mathrm{V}_{\mathrm{n}-1}$ where$V_i=\{v_{i,1},v_{i,2},\dots,v_{i,m}\}$, for $i=1,2,\dots,n-1$.

(i) The harmonic index of wheelbarbell graph for$\mathrm{n}\geq 4$and$\mathrm{m}\geq 2$.

$$H(G)=\frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=2}^{m}\sum_{k=2,k\neq j}^{m}\frac{2}{d(v_{i,j})+d(v_{i,k})}+\sum_{i=1}^{n-1}\sum_{j=2}^{m}\frac{2}{d(v_{i,1})+d(v_{i,j})}$$
$$+\sum_{i=1}^{n-1}\frac{2}{d(v_{i,1})+d(v_{i+1,1})}+\sum_{i=1}^{n-1}\frac{2}{d(u)+d(v_{i,1})},$$

where the addition $\mathrm{i}+1$ in suffix is taken modulo $\mathrm{n}-1$.

$$=\frac{(n-1)(m-1)(m-2)}{2}\left(\frac{2}{m-1+m-1}\right)+(n-1)(m-1)\left(\frac{2}{m+2+m-1}\right)$$
$$+(n-1)\left(\frac{2}{m+2+m+2}\right)+(n-1)\left(\frac{2}{m+2+n-1}\right).$$
$$=(n-1)\left[\frac{m-2}{2}+\frac{2(m-1)}{2m+1}+\frac{1}{m+2}+\frac{2}{m+n+1}\right].$$

(ii)**Case 1:** When $\mathrm{n}=4$

The eccentricity for the vertices in the graph G is given by

$$e(u)=2,$$

$e(v_{i,1})=2$ for $1\leq \mathrm{i}\leq 3$ and

$e(v_{i,j})=3$ for $1\leq \mathrm{i}\leq 3$ and $\mathrm{j}=2,3,\dots,\mathrm{m}$.

The eccentric harmonic index of wheelbarbell graph is

$$H_e(G)=\frac{1}{2}\sum_{i=1}^{3}\sum_{j=2}^{m}\sum_{k=2,k\neq j}^{m}\frac{2}{e(v_{i,j})+e(v_{i,k})}+\sum_{i=1}^{3}\sum_{j=2}^{m}\frac{2}{e(v_{i,1})+e(v_{i,j})}$$
$$+\sum_{i=1}^{3}\frac{2}{e(v_{i,1})+e(v_{i+1,1})}+\sum_{i=1}^{3}\frac{2}{e(u)+e(v_{i,1})},$$

where the addition $\mathrm{i}+1$ in suffix is taken modulo 3.

$$=\frac{3(m-1)(m-2)}{2}\left(\frac{2}{3+3}\right)+3(m-1)\left(\frac{2}{2+3}\right)+3\left(\frac{2}{2+2}\right)+3\left(\frac{2}{2+2}\right).$$
$$=\frac{(m-1)(m-2)}{2}+\frac{6(m-1)}{5}+3.$$

Case 2: When $\mathrm{n}\geq 5$

The eccentricity for the vertices in the graph G is given by

$$e(u)=2,$$

$e(v_{i,1})=3$ for $i=1,2,\dots,n-1$and

$e(v_{i,j})=4$for $i=1,2,\dots,n-1$ and $j=2,3,\dots,m$.

The eccentric harmonic index of wheelbarbell graph is

$$H_e(G)=\frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=2}^{m}\sum_{k=2,k\neq j}^{m}\frac{2}{e(v_{i,j})+e(v_{i,k})}+\sum_{i=1}^{n-1}\sum_{j=2}^{m}\frac{2}{e(v_{i,1})+e(v_{i,j})}$$
$$+\sum_{i=1}^{n-1}\frac{2}{e(v_{i,1})+e(v_{i+1,1})}+\sum_{i=1}^{n-1}\frac{2}{e(u)+e(v_{i,1})},$$

where the addition $\mathrm{i}+1$ in suffix is taken modulo $n-1$.

$$= \frac{(n-1)(m-1)(m-2)}{2}\left(\frac{2}{4+4}\right) + (n-1)(m-1)\left(\frac{2}{3+4}\right)$$
$$+(n-1)\left(\frac{2}{3+3}\right) + (n-1)\left(\frac{2}{2+3}\right).$$
$$= (n-1)\left[\frac{(m-1)(m-2)}{8} + \frac{2(m-1)}{7} + \frac{11}{15}\right].\square$$

Note 3.2. In Theorem 3.1, if we take $m = 1$ then the wheelbarbell graph $G = WB_{\underbrace{1,1,\dots,1}_{(n-1)\text{times}}}$ is isomorphic to wheel W_n.□

Theorem 3.3. For $n \geq 4$, harmonic index and eccentric harmonic index of wheel is given by

(i) $H(W_n) = (n-1)\left[\frac{1}{3} + \frac{2}{n+2}\right]$

(ii) $H_e(W_n) = \begin{cases} 6 & : n = 4 \\ \frac{7(n-1)}{6} & : n \geq 5 \end{cases}$

Proof. Let the $V(G) = \{u, v_1, \dots, v_{n-1}\}$ (see Figure 1 for W_6). For $n \geq 4$, the harmonic index of W_n is

$$H(W_n) = \sum_{i=1}^{n-1} \frac{2}{d(v_i) + d(v_{i+1})} + \sum_{i=1}^{n-1} \frac{2}{d(u) + d(v_i)},$$

where the addition $i + 1$ in the suffix is taken modulo $n - 1$.

$$= (n-1)\left(\frac{2}{3+3}\right) + (n-1)\left(\frac{2}{n-1+3}\right).$$
$$= (n-1)\left[\frac{1}{3} + \frac{2}{n+2}\right].$$

The eccentric harmonic index of W_4 is $H_e(W_4) = 6$ and for $n \geq 5$, the eccentricity of the vertices in W_n are given by

$e(u) = 1$, and

$e(v_1) = 2$ for $i = 1, 2, \dots, n-1$.

The eccentric harmonic index of W_n for $n \geq 5$ is

$$H_e(W_n) = \sum_{i=1}^{n-1} \frac{2}{e(v_i) + e(v_{i+1})} + \sum_{i=1}^{n-1} \frac{2}{e(u) + e(v_i)},$$

where the addition $i + 1$ in the suffix is taken modulo $n - 1$.

$$= (n-1)\left(\frac{2}{2+2}\right) + (n-1)\left(\frac{2}{1+2}\right).$$
$$= \frac{7(n-1)}{6}.$$

Thus $H_e(W_n) = \begin{cases} 6 & : n = 4 \\ \frac{7(n-1)}{6} & : n \geq 5 \end{cases}$□

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